Institute of Mathematical Sciences

MAGNETO-FLUID DYNAMICS DIVISION

MF-8 NYO-9370

THEORY OF CUSPED GEOMETRIES
IV. PARTICLE LOSSES IN CROSSED FIELDS

by

John Killeen

November 15, 1960

AEC RESEARCH AND DEVELOPMENT REPORT

NEW YORK UNIVERSITY

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PHYSICS AND
MATHEMATICS

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ABSTRACT

The loss of particles through a cusp of a particular containment geometry utilizing cusped magnetic field lines is considered. A velocity space loss criterion analogous to the loss cone in the mirror machine is derived. The effect of a uniform longitudinal magnetic field perpendicular to the containing field is considered and a loss criterion is derived. The effect of the longitudinal field is to decrease cusp losses.

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PREFACE

This report presents some calculations done during the summer of 1956. The results were presented at the Project Sherwood meeting held in Berkeley in 1957. A short summary of the calculations appears in the proceedings of that meeting [1].

The idea of using a generalized adiabatic invariant in the calculation was suggested by Dr. Harold Grad [2,3,4]. I should like to acknowledge many helpful discussions with Dr. J. Berkowitz and Dr. George Morikawa.

THEORY OF CUSPED GEOMETRIES

IV. Particle Losses in Crossed Fields

I. Introduction.

Consider the three-dimensional cusped geometry shown in Figure 1. In this configuration plasma is assumed to be contained by a magnetic field with field lines as shown and there is a sharp separation between plasma and vacuum field. There exists an equilibrium solution to the hydro-

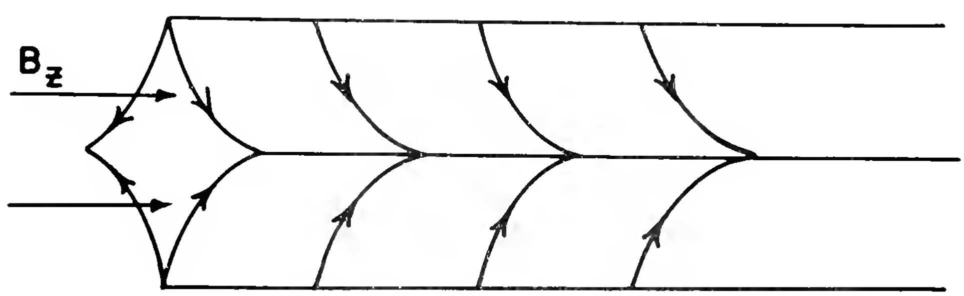


FIG. I

magnetic free surface problem and it is well known that this geometry is hydromagnetically stable [3,6,7].

In order to consider particle losses we must modify the sharp boundary between plasma and field and assume instead a bounding layer in which plasma and field are mixed.

It is in this layer that particles leaving the interior are turned around by the field and returned to the interior. With this modification there are then particle losses through the cusps. The present paper gives a report on an analysis of these cusp losses.

We first consider the case where the interior region is field free; the particles move towards the cusps as shown in Figure 2 and are reflected or lost according to some criterion to be determined. The z-axis is out of the plane of the

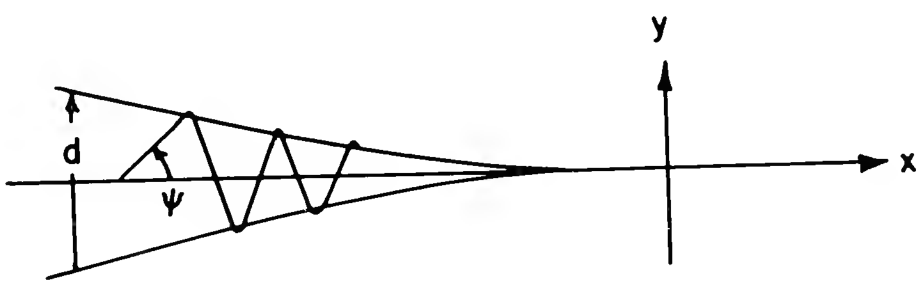


FIG. 2

paper. Let the velocity of a particle be $\vec{v} = (u,v,w)$ where u,v,w are the x,y,z components of velocity in the coordinate system shown. We define

$$\psi = \tan^{-1} \frac{v}{u} \tag{1}$$

Consider this angle as the particle enters the bounding layer at a point where the separation is d. We shall determine a critical angle $\psi^*(d)$ such that for a particle with $\psi(d) > \psi^*(d)$ it is reflected and for $\psi(d) \leq \psi^*(d)$ it is lost. This velocity space loss region is analogous to the loss

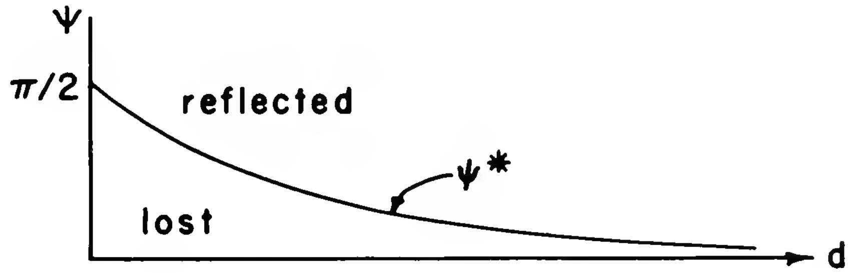


FIG. 3

cone in the mirror machine. With such a loss criterion we can calculate loss rates through the cusps.

We consider now the case of a B_z in the interior region only. Particles move in circular orbits in the x-y plane, i.e., they spiral about the B_z lines. Only those particles which are near enough to the boundary to penetrate the layer will move towards a cusp. There

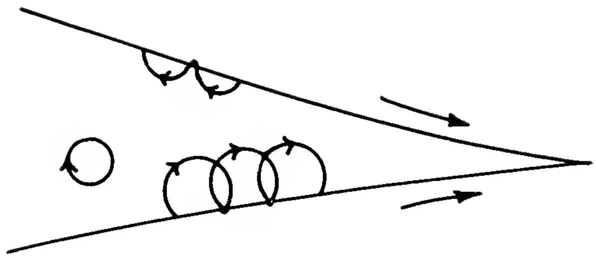


FIG. 4

is no particle reflection by the converging field lines until the particle is close enough to the cusp to penetrate both sides. It can then be reflected or lost according to a criterion to be determined.

The method used in this report to calculate losses is to make use of two constants of the charged particle motion. The first is the speed which is an absolute invariant of the motion; the second is an adiabatic invariant of the motion. If we consider the particle motion in the neighborhood of the cusp we see that a particle penetrating this layer will be turned around by the magnetic field and re-enters the plasma. It then penetrates the layer on the other side and is turned around, etc. It continues this motion towards the cusp until it is either reflected or lost. Now if we consider one complete cycle of this motion

and assume that in the space of one such cycle the bounding surfaces of the plasma are parallel, then the motion is periodic in velocity space. In configuration space it is of course periodic in one direction by definition and is uniformly displaced in the other two directions. There is a periodic motion in phase space which can be considered. The integral $\oint \vec{p} \cdot d\vec{q}$ can then be evaluated around this closed curve in phase space, where $\vec{p}$ and $\vec{q}$ are the Hamiltonian variables of the system i.e. $\vec{p} = m\vec{v} + \frac{e}{c}\vec{A}$, where $\vec{A}$ is the magnetic vector potential. Now if we assume that the separation between the bounding surfaces decreases gradually toward the cusp, then we assert that the integral $\oint \vec{p} \cdot d\vec{q}$ is an invariant of the motion.

We shall use this approximate constant of the motion and the speed to determine a reflection criterion for particles and loss fractions in a manner analogous to the calculation of the same quantities in a magnetic mirror.

2. The Case of no B_z in the Plasma i.e. $B_z = 0$

We consider the geometry as shown in Figure 5.

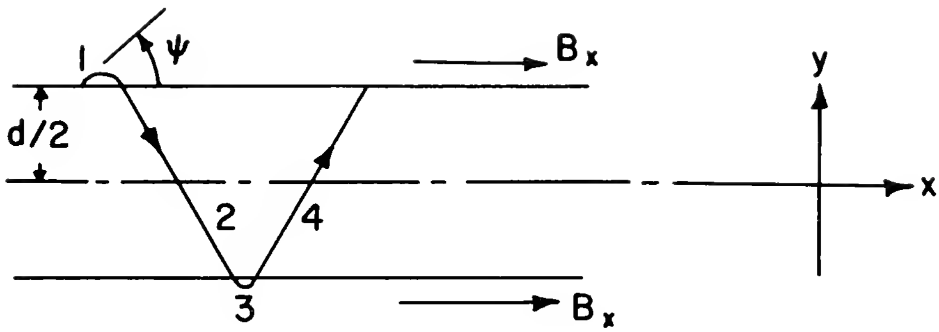


FIG. 5

The z-axis is out of the plane of the paper. We assume the bounding surfaces to be parallel with separation given by d. In the layer we assume $\vec{E} = 0$ and $B_x = B =$ constant, $B_y = B_z = 0$. In the plasma $\vec{E} = 0$ and $\vec{B} = 0$. The equations of motion of a particle in the layer are then

$$m\dot{u} = o$$

(2.1) $$m\dot{v} = e/c \; wB$$

$$m\dot{w} = -\left(^{e}/c\right) v \; B$$

if we let

(2.2) $$\Omega = \frac{eB}{mc}$$

and introduce the new independent variable

$$\theta = \Omega t \tag{2.3}$$

then the equations (2.1) become

$$\begin{aligned} u' &= 0 \\ v' &= w \\ w' &= -v \end{aligned} \tag{2.4}$$

where the prime indicates $d/d\theta$.

The solutions of equations (2.4) are

$$\begin{aligned} u &= u_o \\ v &= w_o \sin\theta + v_o \cos\theta \\ w &= -v_o \sin\theta + w_o \cos\theta \end{aligned} \tag{2.5}$$

where the initial conditions are $u(0) = u_o$, $v(o) = v_o$, $w(o) = w_o$.

The particle trajectories in the layer are then given by

$$\begin{aligned} x &= (u_o/\Omega)\theta + x_o \\ y &= (v_o/\Omega)\sin\theta + (w_o/\Omega)(1 - \cos\theta) + y_o \\ z &= (w_o/\Omega)\sin\theta - (v_o/\Omega)(1 - \cos\theta) + z_o \end{aligned} \tag{2.6}$$

where $x(0) = x_o$, $y(0) = y_o$, $z(0) = z_o$.

In the plasma there are no forces so the particle moves in a straight line (neglecting collisions) with constant velocity.

We now consider the trajectory shown in Figure 5. The particle enters the layer at the point $(x_o, d/2, z_o)$ with velocity components (u_o, v_o, w_o) at $t = \theta = 0$. At the end of path 1 the particle re-enters the plasma at the point $(x_1, d/2, z_1)$ with velocity $(u_o, -v_o, w_o)$ where the time of path 1 is given by

$$T_1 = \theta_1/\Omega = 2/\Omega \tan^{-1} \left(-\frac{v_o}{w_o}\right)$$

Hence we have

$$x_1 - x_o = u_o T_1$$

and

$$z_1 - z_o = - 2 v_o/\Omega$$

At the end of path 2 the particles enters the layer at $(x_2, - \frac{d}{2}, z_2)$ with velocity $(u_o, -v_o, w_o)$ and the time of path 2 is

$$T_2 = \theta_2/\Omega = d/v_o$$

Then $x_2 - x_1 = u_o T_2$ and $z_2 - z_1 = w_o T_2$. The particle re-enters the plasma at the end of path 3 at the point $(x_3, -\frac{d}{2}, z_3)$ with velocity (u_o, v_o, w_o). The time of path 3 is given by

$$T_3 = \theta_3/\Omega = 2/\Omega \tan^{-1}(v_o/w_o)$$

We have $x_3 - x_2 = u_o T_3$ and $z_3 - z_2 = 2\, v_o/\Omega$
A complete cycle of the motion is completed at the end of path 4 with position $(x_4, +\frac{d}{2}, z_4)$ and velocity (u_o, v_o, w_o). The time of path 4 is

$$T_4 = \theta_4/\Omega = d/v_o$$

then $x_4 - x_3 = u_o T_4$ and $z_4 - z_3 = w_o T_4$. We see that $x_4 - x_o = u_o[2\, d/v_o + 2\pi/\Omega]$ since $\theta_1 + \theta_3 = 2\pi$. We also have

$$z_4 - z_o = w_o\, 2d/v_o$$

For this trajectory just considered we wish to evaluate the integral $\oint \vec{p} \cdot d\vec{q}$ around a closed path in phase space. We note that

$$\vec{p} = m\vec{v} + e/c\ \vec{A}$$

where $\vec{A}$ is the vector potential. We have that $\vec{B} = (B,0,0)$ in the layer and $\vec{B} = (0,0,0)$ in the plasma. $\vec{A}$ is determined by $\vec{B} = \text{curl}\ \vec{A}$ and is satisfied by $\vec{A} = (0,0,A_z)$ where $A_z(y)$ is given by

$$(2.7) \qquad A_z = \begin{cases} B(y + d/2) & ; \quad y < -d/2 \\ 0 & ; \ -d/2 < y < d/2 \\ B(y - d/2) & ; \quad y > d/2 \end{cases}$$

We shall now consider this motion in the various phase planes. The x-motion is uniform with constant velocity u_o. Consequently, when this constant motion is subtracted from the x-motion the closed path in phase space is a point and the x-component of the integral is zero. The y-motion is periodic. The y-component of the integral is then

$$\oint p_y dq_y = \oint mv dy = m/\Omega \left\{ \int_o^{\theta_1} v^2 d\theta + \int_o^{\theta_2} v^2 d\theta + \int_o^{\theta_3} v^2 d\theta + \int_o^{\theta_4} v^2 d\theta \right\}$$

The z-motion is characterized by a displacement $(z_4 - z_o)$ in time

$$2\ d/v_o + 2\ \pi/\Omega$$

so the constant displacement velocity is

$$\bar{w} = \frac{w_o 2d/v_o}{[2d/v_o + 2\pi/\Omega]} \tag{2.8}$$

If this constant motion is then subtracted from the z-motion, the closed path in the phase plane is then a circle. The z component of the integral is then given by

$$\oint p_z dq_z = m\oint[(w - \bar{w}) + e/mcA_z](w - \bar{w})dt$$

$$= m/\Omega \left\{ 2\int_0^{\Theta_2} (w - \bar{w})^2 d\Theta + \int_0^{\Theta_3} [w - \bar{w}) + \Omega(y + d/2)](w-\bar{w})d\Theta \right.$$

$$\left. + \int_0^{\Theta_1} [(w - \bar{w} + \Omega(y - d/2)](w - \bar{w})d\Theta \right\}$$

Now let

$$\mu = \frac{1}{m} [\oint p_y dq_y + \oint p_z dq_z]$$

Making use of (2.5), (2.6), (2.8), and $\Theta_1, \Theta_2, \Theta_3$ we evaluate the above expression for μ which gives

$$\mu = 2v_o d + \pi/\Omega(v_o^2 + w_o^2) \tag{2.9}$$

This expression is a generalization of the magnetic moment for the motion considered, in fact the second term is the magnetic moment of the particle in the surface layer.

In order to use the expression (2.9) we consider d to be a gradually decreasing function of x with $d(0) = 0$. We assert that

$$(2.10) \qquad \mu(d,v,w) = \text{constant}$$

for a particle moving in this region, where v and w are the y and z components of velocity of the particle as it enters the surface layer at a point where the separation is d. We also have

$$(2.11) \qquad V^2 = u^2 + v^2 + w^2 = \text{constant}$$

From (2.10) and (2.11) we see that as a particle moves toward the cusp, i.e. as d decreases, the x-component of velocity, u, decreases. When the particle turns around and starts back, $u = 0$, and μ is given by

$$\mu \mid_{u=o} = 2vd + (\pi/\Omega)\, v^2$$

For reflection within the device we must have $d > 0$ when $u = 0$. This condition is satisfied for particles with

(2.12) $$\mu > (\pi/\Omega) V^2$$

From equation (2.9) we have

$$\frac{v}{V} = -\delta/\pi + ((\delta/\pi)^2 + \frac{\mu}{(\pi/\Omega)V^2} - (w/V)^2)^{1/2}$$

where $\delta = \frac{\Omega d}{V}$, the dimensionless variables expressing the separation. If we let $\mu^* = (\pi/\Omega)V^2$, the critical value of μ for reflection, then the above equation gives

(2.13) $$(v/V)^* = -\delta/\pi + [(\delta/\pi)^2 + 1 - (w/V)^2]^{1/2}$$

Now we define

(2.14) $$\psi = \tan^{-1}\frac{v}{u} = \sin^{-1}\frac{v}{(u^2 + v^2)^{1/2}}$$

ψ is the angle in the x-y plane that the velocity vector makes with the x-axis. Then from (2.13) we have

$$(2.15) \qquad \psi^* = \sin^{-1}\left[\frac{-\delta/\pi + [(\delta/\pi)^2 + 1 - (w/V)^2]^{1/2}}{(1 - (w/V)^2)^{1/2}}\right]$$

Equation (2.15) defines the critical angle ψ^*. At a point with separation d, particles entering the layer with velocity (u,v,w) are reflected before reaching the cusp if $\psi > \psi^*$.

The maximum function $\psi^*(d)$ is the one for which $w = o$. If we consider only this most conservative case, then this is the same as considering the two dimensional problem in the x-y plane. Thus, the critical angle in the two dimensional case is

$$(2.16) \qquad \psi^* = \sin^{-1}\left[-\delta/\pi + ((\delta/\pi)^2 + 1)^{1/2}\right]$$

In a report on particle losses by Dr. J. Berkowitz [5], an expression for $\sin \psi^*$ is derived by a different method. Both methods yield the same expression for $\sin \psi^*$, however.

In Figure 7, $\sin \psi^*$ is plotted against δ. To compute a loss fraction, consider a plane perpendicular to the x-axis at a point where the separation is equal to d. If we consider particles with a fixed speed, v, and an isotropic distribution of velocities then the number of particles passing through this plane is proportional to d, i.e.

$$N \sim V^2 \int_{-d/2}^{d/2} dy \int_{o}^{\pi/2} \cos\psi d\psi \sim V^2 d$$

Here we take the angle ψ to be the angle the velocity vector makes with the x-axis in the x-y plane as the particle passes through the perpendicular plane. If we assume that particles passing through this plane with angle ψ, then enter the surface layer at approximately the same angle ψ and approximately the same separation d, then the loss fraction of particles passing through this plane is given by

$$L(d) = \frac{\int_{-d/2}^{d/2} dy \int_{0}^{\psi^{*}(d)} \cos\psi d\psi}{\int_{-d/2}^{d/2} dy \int_{0}^{\pi/2} \cos\psi d\psi}$$

Hence

$$L(d) = \sin \psi^{*}(d). \tag{2.17}$$

The function $\sin \psi^{*}$ os plotted in Fig. 7 for the case $w = 0$. For large d, we have

$$L(d) \approx 1.5 \frac{V}{\Omega} \frac{1}{d} \tag{2.18}$$

This gives an effective "hole" for losses of diameter $1.5 \frac{V}{\Omega}$.

Instead of considering the loss fraction for the case $w \equiv 0$, we can compute it for the three dimensional case as well. Let

$$u = V \cos \psi \sin \theta$$

$$v = V \sin \psi \sin \theta$$

$$w = V \cos \theta$$

then equation (2.15) becomes

$$\sin \psi^* = \frac{1}{\sin \theta} \left\{ - \delta/\pi + \left[(\delta/\pi)^2 + \sin^2 \theta \right]^{1/2} \right\} .$$

If we assume an isotropic distribution of velocities then the loss fraction of particles passing through a plane perpendicular to the x-axis at a point where the separation is equal to d is given by

$$L(d) = \frac{\int_{-d/2}^{d/2} dy \int_0^{\pi} \sin^2\theta \, d\theta \int_0^{\psi^*(d,\theta)} \cos \psi \, d\psi}{\int_{-d/2}^{d/2} dy \int_0^{\pi} \sin^2\theta \, d\theta \int_0^{\pi/2} \cos \psi \, d\psi} .$$

Using the expression for $\sin \psi^*$ given above we have

$$L(d) = 2/\pi \left\{ - \delta/\pi + \left[(\delta/\pi)^2 + 1 \right] \sin^{-1} \left[(\delta/\pi)^2 + \right]^{-1/2} \right\} .$$

This expression for L(d) is slightly smaller than the result for $w \equiv 0$ given by equations (2.17) and (2.16).

3. <u>The Case of a B_z in the Plasma.</u>

We now consider the configuration of field and plasma shown in Fig. 6.

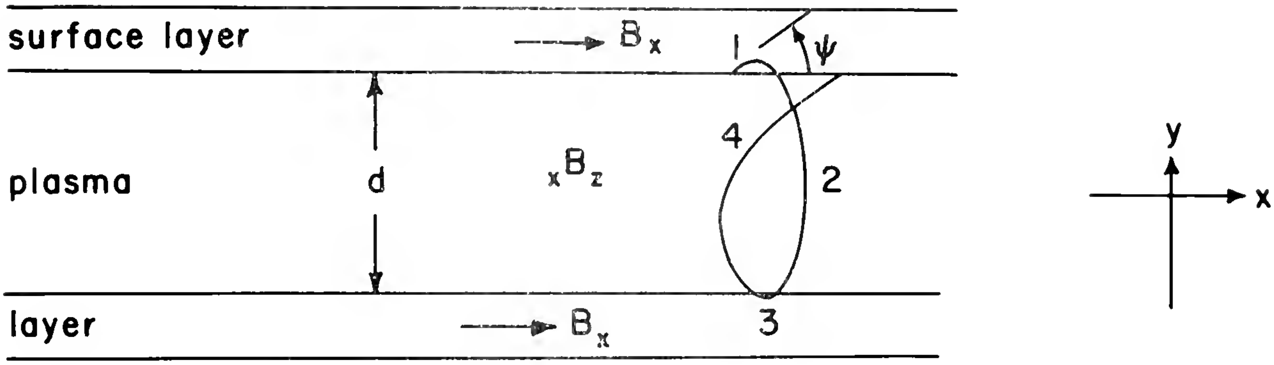


FIG. 6

The component B_z is perpendicular to the plane of the paper, point out. The bounding surfaces are parallel with separation d. In the layer we assume $\vec{E} = 0$ and B_x = const., $B_y = B_z = 0$. In the plasma $\vec{E} = 0$ and B_z = const. = 0, and $B_x = B_y = 0$. The equations of motion of a particle in the plasma are

$$
\begin{aligned}
m\dot{u} &= (e/c)\, v\, B_z \\
m\dot{v} &= -(e/c)\, u\, B_z \qquad (3.1)\\
m\dot{w} &= 0
\end{aligned}
$$

If we let

$$(3.2) \qquad \omega = \frac{eB_z}{mc}$$

and introduce the new independent variable

$$(3.3) \qquad \theta = \omega t$$

then the equations (3.1) become

$$(3.4) \qquad \begin{aligned} u' &= v \\ v' &= -u \\ w' &= 0 \end{aligned}$$

where the prime denotes $\frac{d}{d\theta}$.

The solutions of equations (3.4) are

$$(3.5) \qquad \begin{aligned} v &= v_o \sin\theta + u_o \cos\theta \\ v &= -u_o \sin\theta + v_o \cos\theta \\ w &= w_o \end{aligned}$$

where the initial conditions are $u(o) = u_o$, $v(o) = v_o$, $w(o) = w_o$. The particle trajectories in the plasma are given by

$$
(3.6) \qquad
\begin{aligned}
x &= (u_o/\omega)\sin\theta + (v_o/\omega)(1 - \cos\theta) + x_o \\
y &= (v_o/\omega)\sin\theta - (u_o/\omega)(1 - \cos\theta) + y_o \\
z &= (w_o/\omega)\,\theta + z_o
\end{aligned}
$$

where $x(o) = x_o$, $y(o) = y_o$, $z(o) = z_o$.
In the plasma the particle has a Larmor radius given by

$$
(3.7) \qquad a = 1/\omega(u^2 + v^2)^{1/2} = 1/\omega(u_o^2 + v_o^2)^{1/2}
$$

The equations of motion of a particle in the surface layer are

$$
(3.8) \qquad
\begin{aligned}
m\dot{u} &= 0 \\
m\dot{v} &= (^e/c)w\, B_x \\
m\dot{w} &= -(^e/c)v\, B_x
\end{aligned}
$$

If we let

$$b = \frac{B_x}{B_z} \tag{3.9}$$

then equations (3.8) can be written

$$
\begin{aligned}
u' &= o \\
v' &= bw \\
w' &= -bv
\end{aligned} \tag{3.10}
$$

where the prime denotes $\frac{d}{d\theta}$.

The solutions of equations (3.10) are

$$
\begin{aligned}
u &= u_o \\
v &= w_o \sin b\,\theta + v_o \cos b\,\theta \\
w &= -v_o \sin b\,\theta + w_o \cos b\,\theta
\end{aligned} \tag{3.11}
$$

The particle trajectories in the layer are

$$
\begin{aligned}
x &= (u_o/\omega)\theta + x_o \\
y &= (v_o/b\omega)\sin b\,\theta + (w_o/b\omega)(1 - \cos b\,\theta) + y_o \\
z &= (w_o/b\omega)\sin b\,\theta - (v_o/b\omega)(1 - \cos b\,\theta) + z_o
\end{aligned} \tag{3.12}
$$

Now consider a trajectory of the type shown in Fig. 6

We assume that we are near enough to the cusp so that

$$a > d \, , \tag{3.13}$$

where a is the Larmor radius of a particle when in the plasma. Condition (3.13) insures that the particle will enter both upper and lower surface layers for particles with $u_o > 0$, i.e. particles moving towards the cusp.

The particle enters the layer at the point $(x_o, +\frac{d}{2}, z_o)$ with velocity (u_o, v_o, w_o). At the end of path 1 the particle enters the plasma at the point $(x_1, \frac{d}{2}, z_1)$ with velocity $(u_o, -v_o, w_o)$. The time of path 1 is

$$T_1 = \theta_1/\omega = 2/b\omega \tan^{-1}(-v_o/w_o).$$

so we have

$$x_1 - x_o = u_o T_1$$

At the end of path 2 the particle enters the lower layer at the point $(x_2, -\frac{d}{2}, z_2)$ with velocity (u_2, v_2, w_o), where

$$\begin{aligned} u_2 &= u_o - \omega d \\ v_2 &= -(2\omega u_o d + v_o^2 - \omega^2 d^2)^{1/2} \end{aligned} \tag{3.14}$$

The time of path 2 is

$$T_2 = \theta_2/\omega = {}^1/\omega \sin^{-1}\left[\frac{-v_o u_2 - u_o v_2}{a^2 \omega^2}\right]$$

Thus the x-displacement is given by

$$x_2 - x_1 = - 1/\omega\,(v_o + v_2)$$

At the end of path 3 the particle enters the plasma at the point $(x_3, -\frac{d}{2}, z_3)$ with velocity $(u_2, - v_2, w_o)$. The time of path 3 is

$$T_3 = \theta_3/\omega = 2/b\omega \tan^{-1}(- v_2/w_o)$$

and the x-displacement is $x_3 - x_2 = u_2 T_3$. At the end of path 4 the cycle is completed at the point $(x_4, + \frac{d}{2}, z_4)$ with velocity (u_o, v_o, w_o). The time of path 4 is

$$T_4 = T_2, \text{ so}$$

$$x_4 - x_3 = - \frac{1}{\omega}(v_o + v_2).$$

In the trajectory we have considered the z motion is uniform with velocity w_o when the particle is in the plasma and differs from this uniform motion when the particle is in the layer. If we make the approximation that the z motion is uniform with constant velocity w_o, we can subtract it out completely when considering the integral

$$\oint \vec{p} \cdot d\vec{q}$$

around a closed curve in phase space. Consequently, we make this approximation and consider the two-dimensional problem in the x-y plane i.e. we set $w \equiv o$. With this assumption we have

$$T_1 \doteq T_3 = \pi/b\omega$$

and

$$x_4 - x_o = \frac{1}{\omega}[\pi/b(u_o + u_2) - 2(v_o + v_2)]$$

The x-motion results in a displacement $x_4 - x_o$ in time $2\pi/b\omega + 2\theta_2/\omega$, so the constant displacement velocity is

(3.15) $$\bar{u} = \frac{x_4 - x_o}{[2\pi/b + 2\theta_2/\omega]}$$

If this constant motion is then subtracted from the x-motion the resulting path in the phase plane will be a closed curve.

The y-motion is periodic. We now wish to evaluate $\oint \vec{p} \cdot d\vec{q}$ around the closed path in phase space of the trajectory considered. We have $\vec{p} = m\bar{v} + e/c\bar{A}$. To determine A we note that $\bar{B} = (B_x, 0, 0)$ in the layer and $\bar{B} = (0, 0, B_z)$ in the plasma. This magnetic field can be derived from the vector potential

$$A = (A_x, 0, A_z)$$

where

(3.16) $$A_x = \begin{cases} 0 & ; \quad y < -d/2 \\ -B_z y & ; \quad -d/2 < y < d/2 \\ 0 & ; \quad y > d/2 \end{cases}$$

and

(3.17) $$A_z = \begin{cases} B_x(y + d/2) & ; \quad y < -d/2 \\ 0 & ; \quad -d/2 < y < d/2 \\ B_x(y - d/2) & ; \quad y > d/2 \end{cases}$$

We then have, neglecting the z motion,

$$\overline{\mu} = \frac{1}{m} \oint \vec{p} \cdot d\vec{q}$$

$$= \frac{1}{\omega} \int_0^{\theta_1} [(u - \overline{u})^2 + v^2] d\theta + \frac{1}{\omega} \int_0^{\theta_3} [(u - \overline{u}]^2 + v^2] d\theta$$

$$+ \frac{1}{\omega} \int_0^{\theta_2} \left\{ [(u - \overline{u}) - \omega y](u - \overline{u}) + v^2 \right\} d\theta$$

$$+ \frac{1}{\omega} \int_0^{\theta_4} \left\{ [(u - \overline{u}) - \omega y](u - \overline{u}) + v^2 \right\} d\theta$$

Now making use of (5), (6), (7), (11), (14), (15) and the expressions for θ_1, θ_2, θ_3, θ_4 we can evaluate $\overline{\mu}$ as follows:

$$\overline{\mu} = \pi/b\omega\, [v_o^2/2 + (u_o - \overline{u})^2] + (u_o - d/2\omega - \overline{u})[(x_2 - x_1) - u\, \theta_2/\omega]$$

$$+ \frac{1}{\omega}[\, 1/2 a^2\omega^2\theta_2 + 1/2(v_o^2 - u_o^2) \sin\theta_2\cos\theta_2 + u_o v_o \sin^2\theta_2]$$

$$+ \pi/b\omega[\, v_2^2/2 + (u_2 - \overline{u})^2] + (u_2 + d/2\omega - \overline{u})[(x_4 - x_3) - \overline{u}\, \theta_4/\omega]$$

$$+ 1/\omega\, [1/2 a^2\omega^2\theta_4 + 1/2(v_2^2 - u_2^2) \sin\theta_4\cos\theta_4 + u_2 v_2 \sin^2\theta_4]$$

$$\overline{\mu} = \pi/b[\, v_o^2/\omega + u_o d] + 1/\omega \left[u_o v_o - (u_o - \omega d)[v_o^2 + \omega d(2u_o - \omega d)]^{1/2} \right]$$

$$+ a^2\omega \sin^{-1}\left[\frac{-v_o(u_2 - \omega d) + u_o[v_o^2 + \omega d(2u_o - \omega d)]^{1/2}}{a^2\omega^2} \right]$$

The above function $\bar{\mu}$ (d, u, v) is the required adiabatic invariant for this type of particle motion. Again we assume that d is a slowly decreasing function of x with $d(0) = 0$. We assert that

$$\bar{\mu}\ (d, u, v) = \text{const.}$$

We have that $a^2\omega^2 = u^2 + v^2 = \text{const}$ for particle motion in the plasma, and since we are considering the case $w \equiv 0$ we can assume $V^2 = u^2 + v^2 = \text{const}$ for the entire motion. We can then set $a^2\omega^2 = V^2$ in the expression for $\bar{\mu}$ where V is the speed of the particle.

When a particle which is moving toward the cusp turns around and starts back $u = 0$ and $\bar{\mu}$ is given by

$$\bar{\mu}\Big|_{u=o} = \pi/b\omega[V^2] + d(V^2 - \omega^2 d^2)^{1/2} + \frac{V^2}{\omega} \sin^{-1} \frac{\omega d}{V}$$

For the above expression to be meaningful we need $V > dw$, but this is equivalent to (3.13) which is assumed. From the above we see that the critical value of $\bar{\mu}$ for reflection within the device is

$$\overline{\mu}^* = \pi/b\omega\, V^2$$

i.e. particles for which $\overline{\mu}>\overline{\mu}^*$ will be reflected. Dividing the equation for $\overline{\mu}$ by $\overline{\mu}^*$ we have

$$\frac{\overline{\mu}}{\overline{\mu}^*} = \left[\left(\frac{v}{V}\right)^2 + \frac{u}{V}\frac{\omega d}{V}\right] + \frac{b}{\pi}\left[\frac{u}{V}\frac{v}{V} - \left(\frac{u}{V} - \frac{d}{V}\right)\left[\left(\frac{v}{V}\right)^2 + \frac{d}{V}\left(\frac{2u}{V} - \frac{d}{V}\right)\right]^{1/2}\right]$$

$$+ \frac{b}{\pi}\sin^{-1}\left[-\frac{v}{V}\left(\frac{u}{V} - \frac{\omega d}{V}\right) + \frac{u}{V}\left[\left(\frac{v}{V}\right)^2 + \frac{\omega d}{V}\left(\frac{2u}{V} - \frac{\omega d}{V}\right)\right]^{1/2}\right]$$

Let $\ell = \frac{d\omega}{V} = \frac{d}{a}$ and note that $\sin\psi = \frac{v}{V}$ and $\cos\psi = \frac{u}{V}$, then if we let $\overline{\mu} = \overline{\mu}^*$ the above equation gives the equation which defines the critical angle ψ^*,

$$1 = \sin^2\psi^* + \ell\cos\psi^* + b/\pi\,\sin\psi^*\cos\psi^*$$

(3.18)

$$- b/\pi(\cos\psi^* - \ell)[\sin^2\psi^2 + \ell(2\cos\psi^2 - \ell)]^{1/2}$$

$$+ \frac{b}{\pi}\sin^{-1}[-\sin\psi^*(\cos\psi^* - \ell) + \cos\psi^*[\sin^2\psi^* + \ell(2\cos\psi^* - \ell)]^{1/2}$$

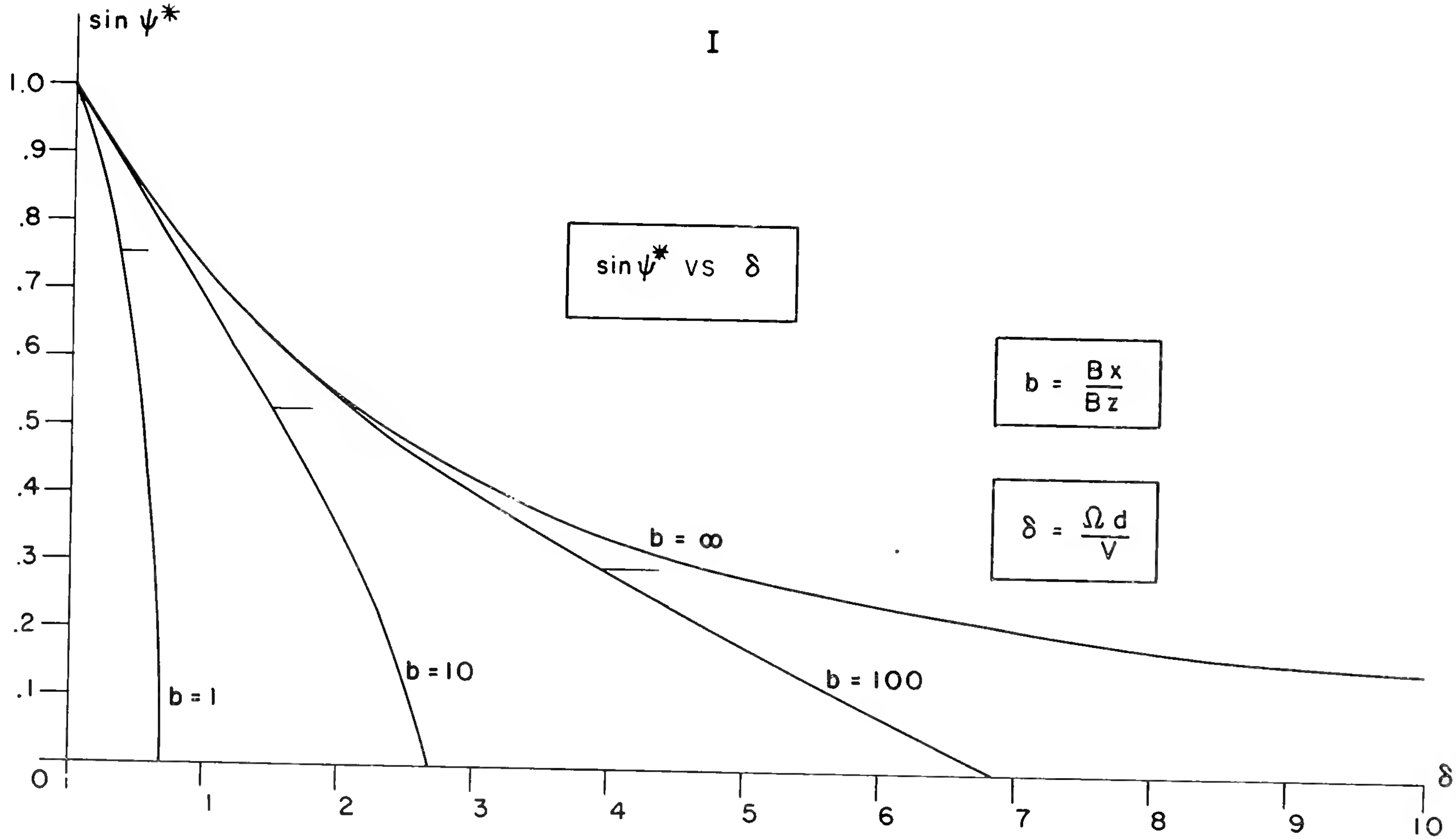
Equation (3.18) defines the critical angle $\psi^*(d)$. At a point on the surface layer with separation d, particles entering the layer at an angle ψ are reflected before reaching the

cusp if $\psi > \psi^*(d)$. Equation (3.18) can be solved numerically for $\sin \psi^*$ as a function of ℓ for various values of b. In order to compare these curves with each other and with the case of no B_z component of the field, it is useful to think in terms of a fixed containing field B_x and plot the curves against the variable $\delta = \frac{\Omega d}{V}$. We note that $\Omega = b\omega$ so

$$(3.19) \qquad \delta = \frac{b\omega d}{V}$$

In figure 7, $\sin \psi^*$, for all the cases, is plotted against δ. In this plot we can see the real effect of the component B_z. In all cases, near the cusp (in the neighborhood of $d = 0$) the curves approach the case of no B_z. Farther away from the cusp the critical angle ψ^* becomes less, particularly for the strong B_z ($b = 1$). In the case of 1% B_z field ($b = 100$) we see that for sufficient distance from the cusp there is even an improvement.

In order to calculate the fraction lost of particles passing through a plane perpendicular to the x-axis, for an arbitrary point on the x-axis (i.e. an arbitrary separation d), we must also consider motion in the region where the particle enters only one side of the bounding layer or neither side.



sin ψ*
I
sin ψ* vs δ
b = Bx/Bz
δ = Ωd/V
b = ∞
b = 1
b = 10
b = 100
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
1
2
3
4
5
6
7
8
9
10
δ

FIG. 7

A particle which does not enter either of the bounding layers makes a circular orbit in the x-y plane; consequently, it does not move towards a cusp and cannot be lost.

Particles that enter only one side of the layer move toward or away from the cusp as shown in Figure 4. The maximum distance that the particle moves away from the layer boundary on each cycle remains approximately the same until the particle hits both sides. If we assume the boundaries are converging straight lines the motion is exactly periodic. Particles of one sign (charge) can only move toward the cusp along one side. For example, in Figure 4, is the direction of B_z is out of the paper, positively charged particles move towards the cusp at the right along the lower side and away from the cusp along the upper side. Consequently, to find the number of particles lost in the cusp we need only to consider particles moving along one side.

Consider a particle making this periodic motion towards the cusp along the lower boundary. Let a particle enter the plasma from the layer at an angle ψ_o.

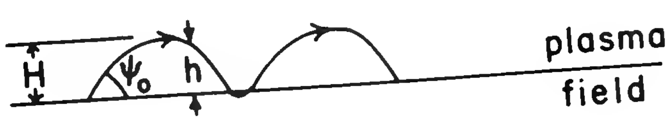


FIG. 8

The distance of the particle from the bounding line is denoted by h, and h is given by

$$h = a[\cos\psi_o(\cos\theta - 1) + \sin\psi_o \sin\theta]$$

(3.20)

$$= a[\cos(\psi_o - \theta) - \cos\psi_o]$$

where $\theta = \omega t$ is measured from when the particle enters the plasma. The maximum distance of the particle is H, given by

(3.21) $$H = a[1 - \cos\psi_o] \text{ .}$$

A particle moving in this manner will start hitting both sides of the bounding layer when the separation, d, decreases such that d = H. From $\psi^*(d)$, determined by equation (3.18), we can evaluate $\psi^*(H)$, and consequently, $\psi^*(\psi_o)$. $\psi^*(\psi_o)$ is independent of a, since we know $\psi^*(\frac{d}{a})$ and thus $\psi^*(\frac{H}{a})$. In Figure 9 we have sin ψ^* vs. sin ψ_o for three values of b. From the relation $\psi^*(\psi_o)$ we can find a value ψ_c^* such that for

$$\psi_o < \psi_c^* \quad \text{then} \quad \psi_o < \psi^*$$

$$\text{and} \qquad \psi_o > \psi_c^* \quad \text{then} \quad \psi_o > \psi^*$$

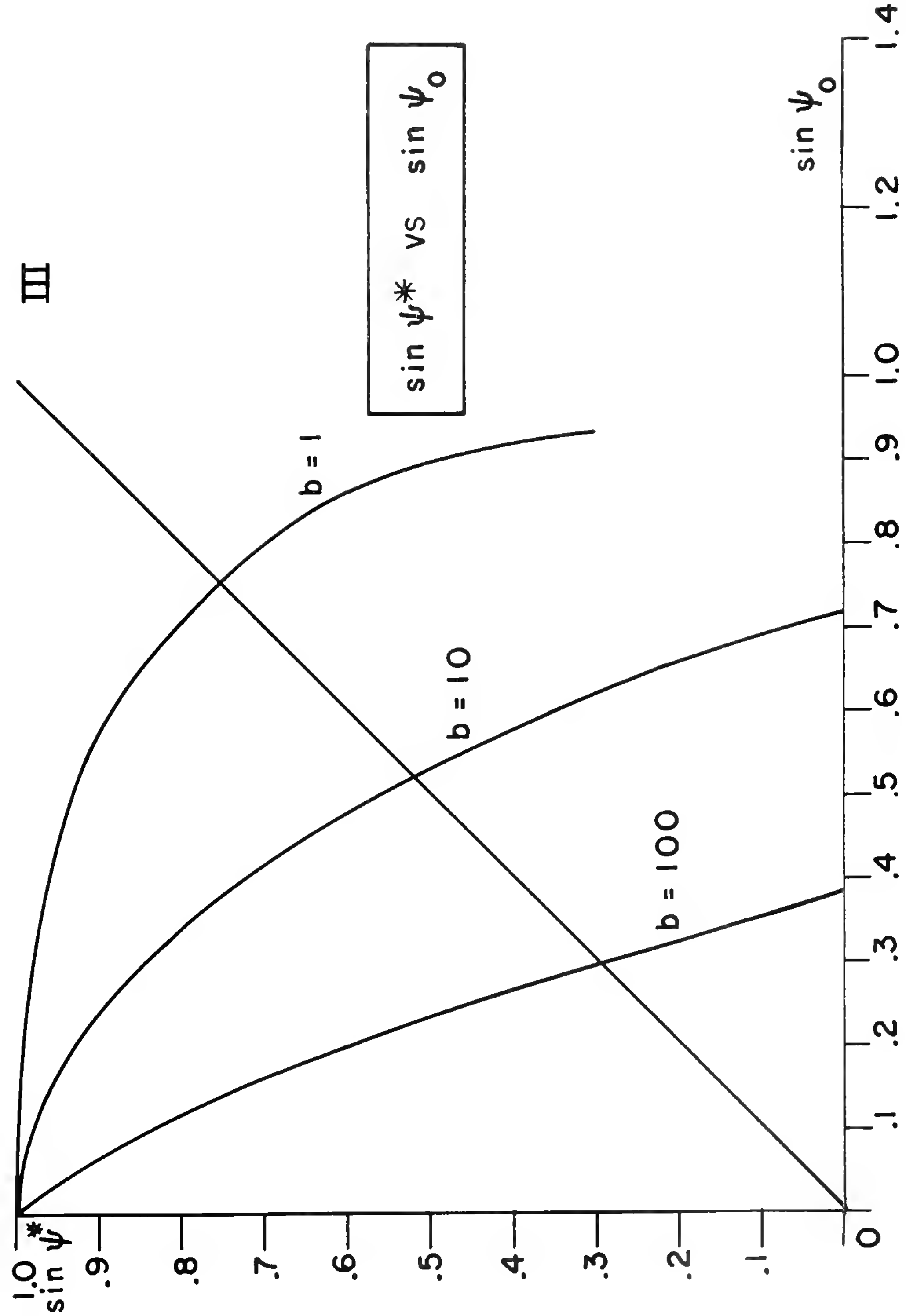
The value $\sin\psi_c^*$ is shown in Figure 9 as the intersection between $\sin\psi^*$ $(\sin\psi_o)$ and the line $\sin\psi^* = \sin\psi_o$.

ψ_c^* is the critical angle for particles moving toward the cusp along one side of the boundary, i.e. for a particle entering or leaving the layer at an acute angle ψ, it will be reflected if $\psi > \psi_c^*$ or lost if $\psi < \psi_c^*$. It is evident that ψ_c^* is independent of d, since the particle motion is periodic along one side.

We wish to calculate the fraction of the particles passing through a plane perpendicular to the X-axis that are lost through the cusp. We consider particles of a fixed speed V, and thus a fixed Larmor radius, a. The perpendicular plane is located as a point where the separation is d. We assume that we are sufficiently far from the cusp so that d>2 a, i.e. the particles can only hit one side of the boundary layer. Then the particles that are lost are only those moving towards the cusp along one side.

The number of particles passing through a plane, assuming a fixed speed and an isotropic distribution of velocities is proportional to

$$V^2 \int_{-d/2}^{d/2} dy \int_{o}^{\pi/2} \cos\psi d\psi = V^2 d$$



III
sin ψ* VS sin ψo
b = 1
b = 10
b = 100
1.0
sin ψ*
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.2
1.4
sin ψo

FIG. 9

In this integral, ψ is the angle that a particle makes with the x-axis at a point in the plane with ordinate y. To find the number of particles lost we need to calculate the integral

$$V^2 \iint_D \cos \psi \, d\psi \, dy$$

where the domain of integration D is given by our established loss criterion. We recall that in the loss criterion the angle ψ considered is the angle that the particle path makes with the layer as it enters or leaves the layer, while in the above integral ψ is the angle the particle path makes with the x-axis passing through a plane at ordinate y. Consequently, we must relate these angles to apply the criterion. Let ψ_o be the angle at the surface and let (Figure 8) ψ be the angle at a point when the distance to the lower side is h. We have from (3.20)

$$h = a[\cos(\psi_o - \theta) - \cos\psi_o]$$

and from $\psi = \tan^{-1} \frac{v}{u}$ and

$$u = v_o \sin \theta + u_o \cos \theta$$

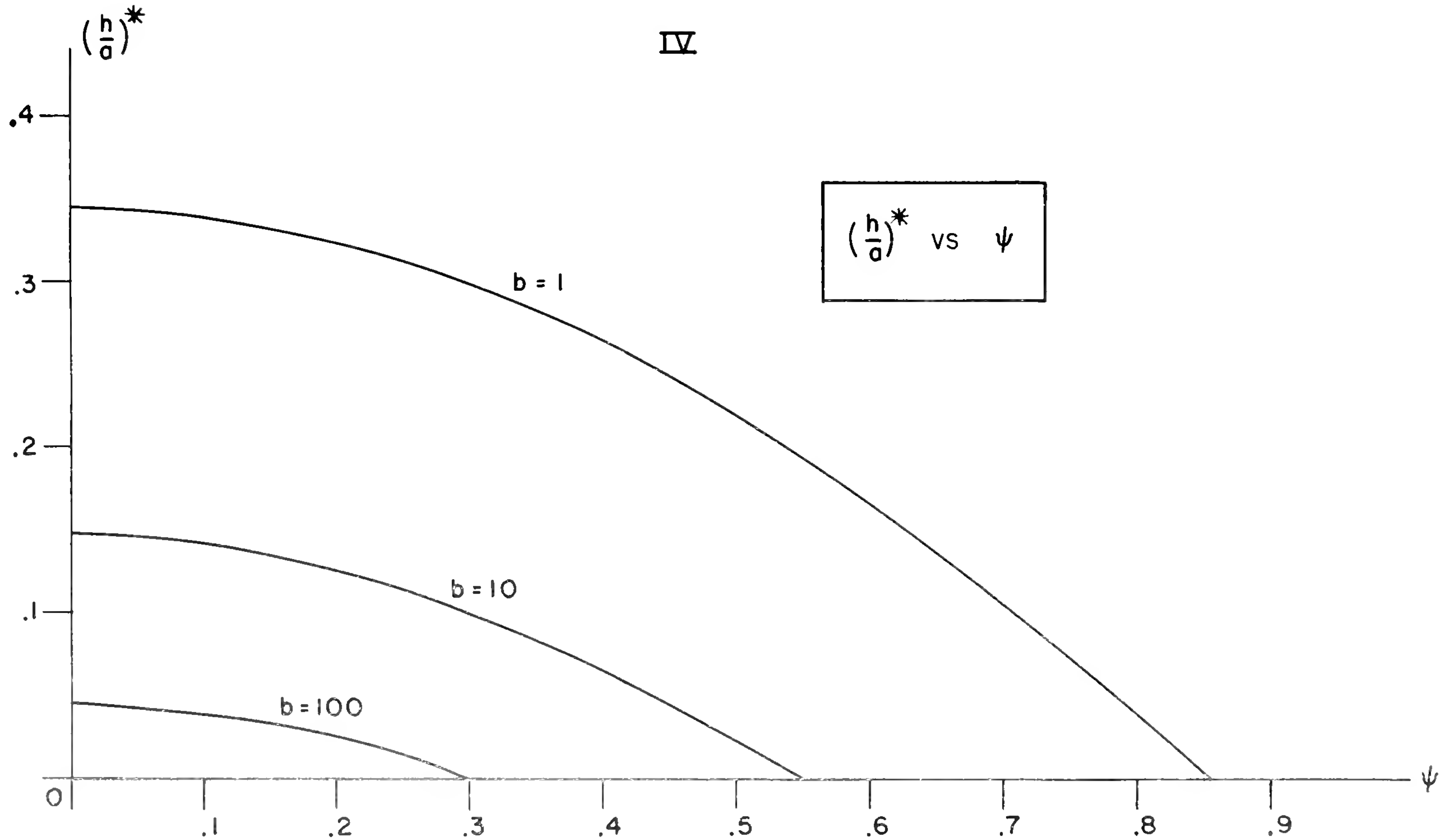


FIG. 10

the relation

$$\cos \psi = \sin \psi_o \sin \theta + \cos \psi_o \cos \theta$$

$$= \cos (\psi_o - \theta) ;$$

consequently, we have

(3.22) $$h = a[\cos \psi - \cos \psi_o]$$

Furthermore if $\psi_o = \psi_c^*$, the critical angle, then

(3.25) $$h^* = a[\cos \psi - \cos \psi_c^*]$$

defines the domain of integration, D. The domains are shown in Figure 10, for the three values of b. We then have

$$\iint_D \cos \psi \, d\psi \, dy = \int_o^{h^*(\psi)} dy \int_o^{\psi_c^*} \cos \psi \, d\psi = \int_o^{\psi_c^*} h^*(\psi) \cos \psi \, d\psi$$

$$= a \int_o^{\psi_c^*} [\cos^2 \psi - \cos \psi_c^* \cos \psi] d\psi = a/2[\psi_c^* - \sin \psi_c^* \cos \psi_c^*]$$

The fraction lost is then given by

(3.24) $$L(d) = 1/2 \, a/d[\psi_c^* - \sin \psi_c^* \cos \psi_c^*]$$

To compare this with the case of no B_z we note that $a = \frac{V}{\omega} = \frac{bV}{\Omega}$, then

$$(3.25) \qquad L(d) = {}^1/_2\left({}^{bV}/_{\Omega d}\right) [\psi_c^*(b) - \sin\psi_c^*(b)\cos\psi_c^*(b)]$$

If we take the values of ψ_c^* given in Figure 9 we have approximately

$$L(d) = .18\,\frac{V}{\Omega}\,\frac{1}{d} \qquad \text{for } b = 1$$

$$L(d) = .52\,\frac{V}{\Omega}\,\frac{1}{d} \qquad \text{for } b = 10$$

$$L(d) = .90\,\frac{V}{\Omega}\,\frac{1}{d} \qquad \text{for } b = 100,$$

which gives an effective "hole" for particle losses of diameter

$$d_L = .18\,\frac{V}{\Omega}, \qquad b = 1$$

$$d_L = .52\,\frac{V}{\Omega}, \qquad b = 10$$

$$d_L = .90\,\frac{V}{\Omega}, \qquad b = 100$$

This is to be compared with

$$d = 1.5\,\frac{V}{\Omega}$$

for the case of no B_z field ($b = \infty$). We see that within the framework of this idealized model the addition of a crossed field produces a marked improvement.

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